ECE 346: Digital Signal Processing — Spring 2021
Soo Min Kwon — Week of February 22, 2021

Recitation #5: Sampling, Aliasing, & Quantizing of Signals

Objective & Outline

The objective of this week's recitation session is to continue working with sampling, aliasing, and quantizing of signals.

- Problems $1-4$: recitation problems
- Problem 5: self-assessment problem

Recitation Notes

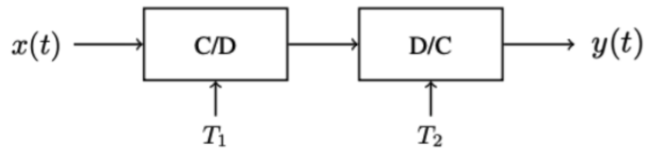
Since quantization is one of the newer topics here, the following are some notes regarding this topic:

- Quantization is how an analog-to-digital converter (ADC) can connect all possible values between some $n_{\max}$ and $n_{\min}$ to 2^B unique numbers, where B is generally pre-specified.
- Uniform quantization is equally dividing our range into 2^B pieces and then assigning our discrete signal to the middle of that piece:
  - Dynamic Range: $R = n_{\max} - n_{\min}$.
  - Quantization Interval: $\delta = R/2^B$
- Quantizing our values can "perturb" our discrete value, producing $\hat{g}[n]$ from $g[n]$ where $\hat{g}[n] \neq g[n]$. The maximum quantization error is defined as

$$\text{Error} = \delta/2 \tag{1}$$

The problems start on the following page.

Problem 1 (Sampling)**.** Consider the following block diagram of a digital signal processing system:



Suppose that the input to this block diagram, $x(t)$, is given by

$$x(t) = \frac{\sin(100\pi t)}{\pi t}, \tag{2}$$

with $T_1 = 1/100$ seconds and $T_2 = 1/150$ seconds.

(a) Provide a labeled plot of the CTFT of $x_p(t)$, where $x_p(t)$ is the impulse sampled signal of $x(t)$, defined as $x_p(t) = \sum_{n=\infty}^{\infty} x(t)\delta(t - nT)$.

(b) Provide a labeled plot of $H_r(j\Omega)$, the ideal reconstruction filter of this system.

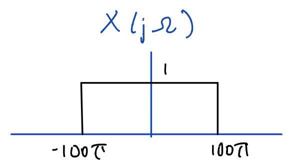
(c) Provide a labeled plot of $Y(j\Omega)$, the CTFT of the output of this system, $y(t)$.

Solution:-

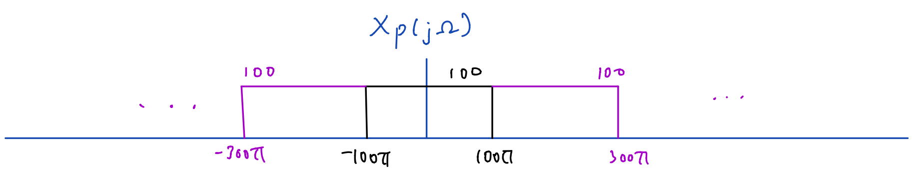
(a) Recall that the CTFT of the impulse sampled signal, say $x_p(t)$, for an arbitrary $x(t)$ is given by

$$X_p(j\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(j(\Omega - n\Omega_T)).$$

This is effectively taking $X(j\Omega)$ and sampling them every Ω_T:



Since $T_1 = 1/100$ seconds, $f_s = 100$ Hz and $\Omega_T = 200\pi$ rad/sec:

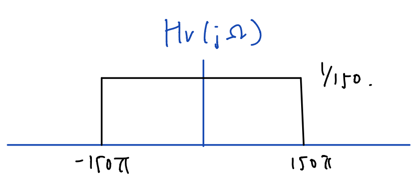


(b) In general, the ideal reconstruction filter, $H_r(j\Omega)$ is defined as

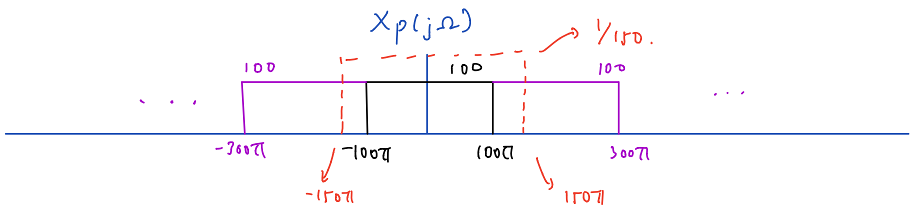
$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \le \pi/T \\ 0, & \text{otherwise} \end{cases}.$$

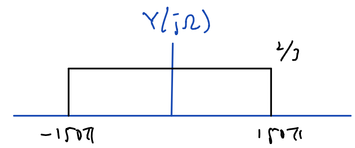
Now given that $T_2 = 1/150$ seconds,

$$H_r(j\Omega) = \begin{cases} 1/150, & |\Omega| \le 150\pi \\ 0, & \text{otherwise.} \end{cases}$$



(c) Now our signal $X_p(j\Omega)$ goes through the filter from previous part:





$$y(t) = \frac{2/3 \sin(150\pi t)}{\pi t}.$$

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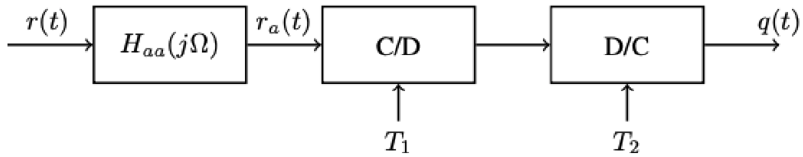
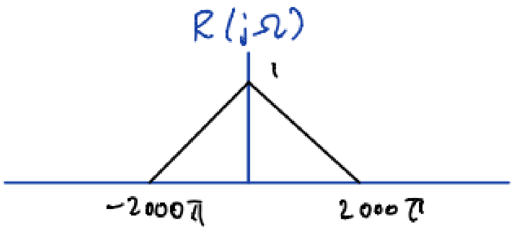
Problem 2 (Sampling)**.** Consider the following block diagram of a digital signal processing system:



with $T_1 = T_2 = 1/1500$ seconds and $r(t) \xleftrightarrow{\text{CTFT}} R(j\Omega)$ given as



(a) Suppose that $H_{aa}(j\Omega)$ was an ideal low-pass filter with a cutoff of Ω_a. What should the value of Ω_a be to ensure that there is no aliasing given T_1 and T_2?

(b) Suppose the cutoff frequency for $H_{aa}(j\Omega)$ was $|\Omega| \leq 1000\pi$:

$$H_{aa}(j\Omega) = \begin{cases} 1, & |\Omega| \leq 1000\pi \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Provide a labeled plot of $R_p(j\Omega)$, the CTFT of the impulse sampled signal of $r_a(t)$.

(c) Provide a labeled plot of $Q(j\Omega)$, the CTFT of the output of this system, $q(t)$.

Solution :-

(a) We are sampling at a frequency of $f_s = 1500$ Hz, or 3000π rad/sec.

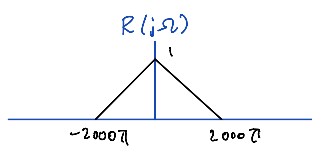
The bandwidth of our signal $R(j\Omega)$ is 2000π. Thus, the Nyquist criteria is not satisfied:

$$3000\pi \ngeq 4000\pi .$$

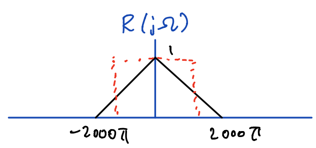
Hence, the cutoff frequency of $H_{aa}(j\Omega)$ should be at least $|\Omega| \le 1500\pi$ so that we get

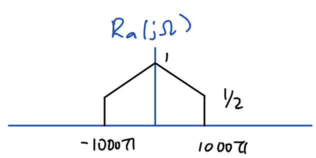
$$3000\pi \geq 3000\pi .$$

(b) We are given

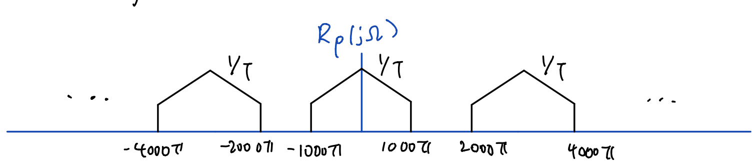


Going through an anti-aliasing filter of $|\Omega| \le 1000\pi$:

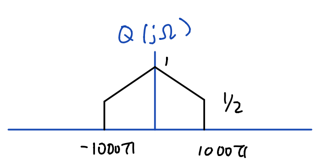




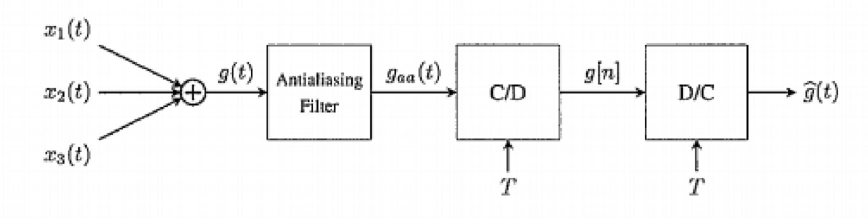
Now sampling this signal at $\Omega_T = 1000\pi$:



(c) Going through an ideal reconstruction filter, we get our signal back:



Problem 3 (Sampling)**.** Consider the following block diagram of a digital signal processing system:



The continuous-time Fourier transforms (CTFTs) of the three signals, $x_1(t)$, $x_2(t)$, and $x_3(t)$ are given as follows:

$$X_1(j\Omega) = \begin{cases} 2, & |\Omega| \leq 40\pi \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

$$X_2(j\Omega) = \begin{cases} 3, & |\Omega| \leq 80\pi \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

$$X_3(j\Omega) = \begin{cases} 2, & |\Omega| \leq 60\pi \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

(a) What is the bandwidth of the continous-time signal $g(t)$? Provide a labeled plot of the CTFT of $g(t)$, $G(j\Omega)$.

(b) Suppose that the value of T was $T = 1/70$ seconds in the block diagram and let the anti-aliasing filter be an ideal low-pass filter with cutoff frequency 70π rad/sec.

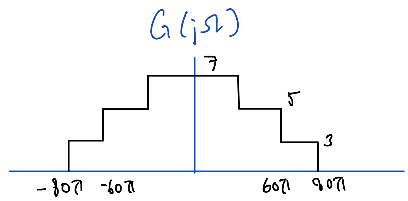
   (a) Provide a labeled plot of the CTFT of $g_{aa}(t)$, $G_{aa}(j\Omega)$.

   (b) Provide a labeled plot of the CTFT of $\hat{g}(t)$, $\hat{G}(j\Omega)$.

(c) Suppose that T was still $T = 1/70$ seconds, but the anti-aliasing filter was not present in the block diagram. Provide a labeled plot of the CTFT of the output signal, $\hat{g}(t)$, for this case.

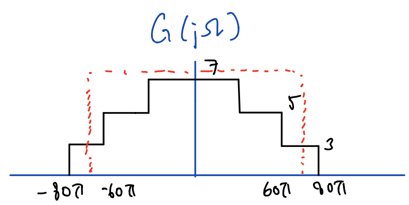
Solution: -

(a) Let's first plot $G(j\Omega) = X_1(j\Omega) + X_2(j\Omega) + X_3(j\Omega)$:

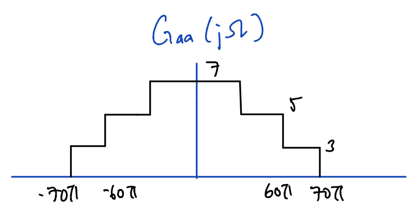


The bandwidth of this signal is clearly 80π rads/sec. ■

(b)



a.

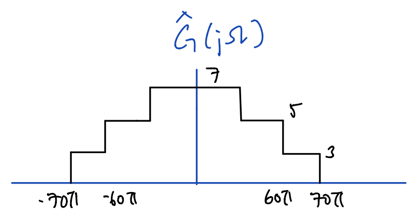


Rather than taking all of the steps like in the previous question, we should ask ourselves: is there aliasing? Well,

$$140\pi \geq 2(70\pi) = 140\pi \checkmark$$

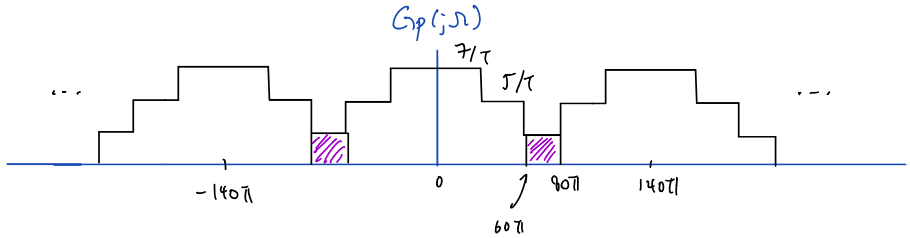
Since the Nyquist criteria is satisfied, we will get back our original (filtered) signal:

b.

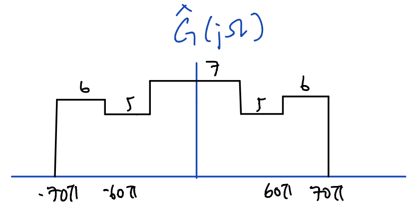


■

(c) We will jump into looking at the impulse sampled signal:



The overlaps will add up and then go through an ideal reconstruction filter with cutoff frequencies $|\Omega| \le 70\pi$:



Problem 4 (Quantization)**.** The Holland Tunnel is equipped with 500 vibration sensing chips. Each one of these chips has an integrated 8-bit, 300 Hz analog-to-digital converter (ADC) that takes an input as voltage signal with maximum voltage of 2.4 V and minimum voltage of 0.8 V.

(a) What is the sampling period for each ADC?

(b) What can be the maximum frequency of the vibration signal being sensed by each chip such that there is no aliasing during sampling?

(c) What is the dynamic range of each ADC?

(d) What is the size of a quantization interval in each ADC?

(e) What is the maximum quantization error that the input vibration signals would incur?

(f) How many samples does each ADC generate per minute?

(g) How much data, in bits/second (bps) would all 500 chips generate?

(h) How much storage capacity, in Bytes, would be needed to store the data generated by all 500 chips in one year?

(i) Suppose it is desired to have the maximum quantization error no larger than 0.25 mV. What should be the minimum bit resolution of the ADCs to achieve this goal?

Solution:-

Many of there answers can be straightforwardly inferred from the problem statement.

(a) $f_s = 300$ Hz.

(b) $f_{max} = f_s/2 = 150$ Hz.

(c) Dynamic range (R) = max voltage − min voltage
= 2.4 V − 0.8 V = 1.6 V.

(d) $\delta = 1.6/2^8 = 6.25$ mV.

(e) Max error = $\delta/2$ = 3.125 mV.

(f) 300 samples/second × 60 seconds/min = 18000 samples/min.

(g) Each chip generates 300 × 8 = 2400 bits per second, so all 500 would generate

$2400 \times 500 = 1.2 \times 10^6$ bps.

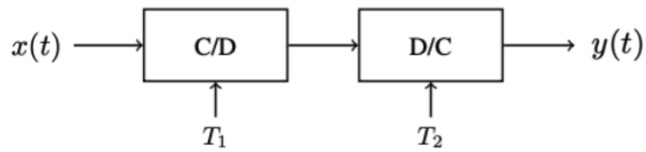
(h) 300 × 500 × # of seconds in a year :

= 4.73 TB.

(i) $\delta/2 \leq 0.25$ mV $\rightarrow$ $\delta \leq 0.5$ mV

$\rightarrow \frac{\text{Range}}{2^B} \leq 0.5$ mV $\rightarrow$ $B \geq \log_2(3200)$
$B \geq 11.64$
$B = 12$.

Problem 5 (Self-assessment)**.** Consider the following block diagram of a digital signal processing system:



Let the CTFT of the continuous-time signal $x(t)$ be defined as

$$X(j\Omega) = \begin{cases} 1, & -200\pi \leq \Omega \leq 0 \\ 2, & 0 < \Omega \leq 200\pi, \end{cases} \tag{7}$$

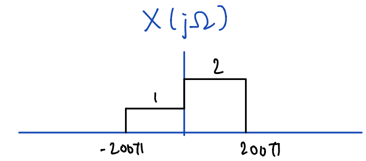
with $T_1 = T_2 = 1/150$ seconds.

(a) Provide a plot $X_p(j\Omega)$ the CTFT of the impulse sampled signal of $x(t)$.

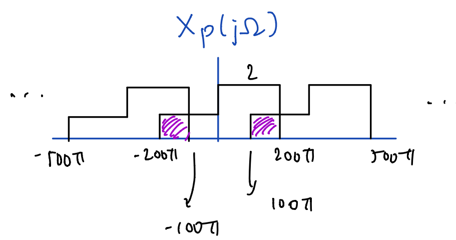
(b) Provide a plot of the CTFT of the output of the system, $Y(j\Omega)$.

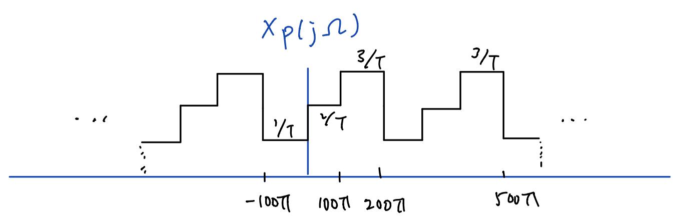
Solution :-

(a) We should first plot $X(j\Omega)$:



Sampling $X(j\Omega)$ every $\Omega_T = 300\pi$, we get





(b) We now send $X_p(j\Omega)$ through an ideal reconstruction filter with cutoff $|\Omega| = 150\pi$: